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# Book Reviews

N. I. AKHIEZER, *Elements of the Theory of Elliptic Functions*, Translations of Mathematical Monographs, Vol. 79, American Mathematical Society, 1990, 237 pp.

This is the long awaited English translation of the 2nd Russian edition (1970; 1st edition 1948) of Akhiezer's book on elliptic functions. Western readers of the cryptic Appendix E of his *Lectures on the Theory of Approximation* (Ungar, New York, 1956) will at last enjoy now the full explanation of the fine technical points. Indeed, some exactly solvable problems of approximation theory ask for a fair knowledge of elliptic functions. The subject could have been thought of as buried in 19th century mathematics, but has experienced a recent revival, as shown in J. M. Borwein and P. B. Borwein's *Pi and the AGM* (Wiley, New York, 1987), and also by books on Ramanujan's (much more cryptic) writings. This recent interest in elliptic functions and related matters triggered perhaps the publishing of this volume. The present book is intended to deliver the science of these functions, without having to introduce elaborate concepts of algebraic geometry. As a result, the main facts about elliptic functions are given in a very constructive way, starting essentially with the double series defining Weierstrass' $\wp$ function. The book is mathematically self-contained, and hard calculations are sometimes needed (in small print). Chapters 1-4 develop the theory up to the differential equation for $\wp$, the expression of any elliptic function in terms of $\wp$ and $\wp'$, the addition theorem, modular and theta functions, and the inversion of elliptic integrals as a by-product. Chapter 5 introduces the probably more popular Jacobi functions. As the author says: "The Jacobi (or Riemann) form ... is very convenient for various computations. The Weierstrass form is almost always preferable for theoretical considerations." Among the most interesting items of the following chapters, one finds special conformal mappings and the transformations of elliptic functions when the periods ratio is multiplied or divided by an integer $n$ (Sect. 40) ($n = 2$: Gauss–Landen transformations, Sects. 38 and 39, related to the arithmetico-geometric mean Sect. 46). Chapter 9 expands the solution of these famous Zolotarev problems about extremal rational functions. This chapter will make the book most valuable for many people in approximation theory and applied mathematics, as the subject is still hot (see J. Todd, "A legacy from E. I. Zolotarev (1847–1878)", in *Approximation Theory and Spline Functions* (S. P. Singh *et al.*, Eds.), pp. 207–245, Reidel, 1984, and *Math. Intelligencer* **10** (1988), 50–53). Chapter 10 (added in the 1970 edition) is about special orthogonal polynomials on two intervals and presents some of the author's own findings. Elementary and intuitive presentations of two-sheeted Riemann surfaces, Abel's theorem in the elliptic case, elliptic coordinates, the Lamé equation, and the Picard and Landau theorems receive a short survey. The Western reader who has longed for this translation for years will perhaps be a little frustrated by the sometimes obscure style which leaves meaningful definitions and propositions to appear almost as casual remarks, especially in the first four chapters (the absolute beginner is informed here that the definition of elliptic functions is to be found in the two last lines of Sect. 3). The Zolotarev polynomial problems are not covered, and the treatment of orthogonal polynomials on several intervals suffers from the limitations of the method (compare with the broader scope of A. I. Aptekarev's "Asymptotic properties of polynomials orthogonal on a system of contours and periodic motions of Toda lattices," *Math. USSR Sb.* **53** (1986), 233–260, for instance). The numerical tables look quite old-fashioned. However, these minor drawbacks will by no means deter the true aficionado.

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